On modular forms of weight $(6n+1)/5$ satisfying a certain differential equation

Masanobu Kaneko

Abstract. We study solutions of a differential equation which arose in our previous study of supersingular elliptic curves. By choosing one fifth of an integer k as the parameter involved in the differential equation, we obtain modular forms of weight k as solutions. It is observed that this solution also relates to supersingular elliptic curves.

1 Introduction

In our previous work [5],[3],[4], we studied various solutions of the specific differential equation

$$(\sharp)_k \qquad f''(\tau) - \frac{k+1}{6}E_2(\tau)f'(\tau) + \frac{k(k+1)}{12}E_2'(\tau)f(\tau) = 0 \qquad \left(' = \frac{1}{2\pi i}\frac{d}{d\tau} = q\frac{d}{dq},\ q = e^{2\pi i\tau}\right),$$

where τ is a variable in the upper half-plane, k a fixed rational number, and $E_2(\tau)$ the "quasimodular" Eisenstein series of weight 2 for the full modular group $SL(2,\mathbf{Z})$:

$$E_2(\tau) = 1 - 24\sum_{n=1}^{\infty}\Big(\sum_{d|n} d\Big)q^n.$$

In [5], we showed that for even $k \geq 4$ with $k \not\equiv 2 \pmod 3$, this differential equation has a modular solution of weight k on $SL(2,\mathbf{Z})$ explicitly describable in terms of the Eisenstein series $E_4(\tau)$ and $E_6(\tau)$, and discussed a connection to supersingular elliptic curves in characteristic p when $k = p-1$. We studied further the modular/quasimodular solutions for other integral or half-integral values of k in [3], [4].

In this paper, we set $k = (6n+1)/5$, one fifth of an integer congruent to 1 modulo 6. We then encounter as solutions modular forms of weight k on the principal congruence subgroup $\Gamma(5)$. Also, modular forms on $\Gamma_1(5)$ arises naturally. In the next section we describe the solutions in terms of fundamental modular forms of weight $1/5$ which already appeared in works of Klein or Ramanujan. The proof is in essence similar to the one given in [3]. In §3, we discuss a relation to supersingular elliptic curves, which is quite analogous to the situation studied in [5] and [8].

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to the present work. The author also learned from him that the differential equation $(\sharp)_k$, in its equivalent form, was already appeared in works of physicists, e.g., [6], [7], and its solutions, at least for small values of k, correspond to particular models in conformal field theory.

2 Main result

Let

$$\begin{aligned}\phi_1 = \phi_1(\tau) &= \frac{1}{\eta(\tau)^{3/5}} \sum_{n\in\mathbf{Z}} (-1)^n q^{(10n+1)^2/40} \\ &= 1 + \frac{3}{5}q + \frac{2}{25}q^2 - \frac{28}{125}q^3 + \frac{264}{625}q^4 + \frac{532}{15625}q^5 + \cdots ,\end{aligned}$$

and

$$\begin{aligned}\phi_2 = \phi_2(\tau) &= \frac{1}{\eta(\tau)^{3/5}} \sum_{n\in\mathbf{Z}} (-1)^n q^{(10n+3)^2/40} \\ &= q^{1/5}(1 - \frac{2}{5}q + \frac{12}{25}q^2 + \frac{37}{125}q^3 - \frac{171}{625}q^4 - \frac{3318}{15625}q^5 + \cdots).\end{aligned}$$

Here, $\eta(\tau)$ is the Dedekind eta function. These forms are of weight $1/5$ (with a suitable multiplier system) on $\Gamma(5)$ and the ring of holomorphic modular forms of weight $\frac{1}{5}\mathbf{Z}$ on $\Gamma(5)$ with this multiplier system is the polynomial ring $\mathbf{C}[\phi_1, \phi_2]$ (a good reference for this is Ibukiyama [2]). Note that these forms are essentially the famous Rogers-Ramanujan functions;

$$\begin{aligned}\phi_1 &= \eta(\tau)^{2/5} q^{-1/60} \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})}, \\ \phi_2 &= \eta(\tau)^{2/5} q^{11/60} \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})}.\end{aligned}$$

Theorem. *Assume $k = (6n+1)/5,\ n = 0, 1, 2, \ldots,\ n \not\equiv 4 \pmod 5$. Then the equation $(\sharp)_k$ has two dimensional space of solutions in $\mathbf{C}[\phi_1, \phi_2]_{wt=k}$, the set of homogeneous polynomials of degree $6n+1$ in ϕ_1 and ϕ_2.*

Example: Here are a basis of solutions for small k:

$$\begin{aligned}
k = \frac{1}{5}: &\qquad \phi_1, \quad \phi_2, \\
k = \frac{7}{5}: &\qquad \phi_1^7 + 7\phi_1^2\phi_2^5, \quad 7\phi_1^2\phi_2^5 - \phi_2^7, \\
k = \frac{13}{5}: &\qquad \phi_1^{13} + 39\phi_1^8\phi_2^5 - 26\phi_1^3\phi_2^{10}, \quad 26\phi_1^3\phi_2^{10} + 39\phi_1^5\phi_2^8 - \phi_2^{13}, \\
k = \frac{19}{5}: &\qquad \phi_1^{19} + 171\phi_1^{14}\phi_2^5 + 247\phi_1^9\phi_2^{10} - 57\phi_1^4\phi_2^{15}, \\
&\qquad 57\phi_1^{15}\phi_2^4 + 247\phi_1^{10}\phi_2^9 - 171\phi_1^5\phi_2^{14} + \phi_2^{19}.
\end{aligned}$$

In general, we have a basis of the form $\phi_1^{n+1}\times$ (polynomial in ϕ_1^5 and ϕ_2^5) and $\phi_2^{n+1}\times$ (polynomial in ϕ_1^5 and ϕ_2^5). Here, we note that ϕ_1^5 and ϕ_2^5 become modular forms of weight 1 on $\Gamma_1(5)=\left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \mid a\equiv d\equiv 1,\ c\equiv 0 \pmod p\right\}$.

In our previous cases treated in [5], [3], [4], all solutions were explicitly described with the aid of hypergeometric series. In the present case, however, a differential equation with four singularities emerges and we are so far not able to write down the explicit formulas of the solutions in general. We can nevertheless prove the theorem by giving the solutions recursively and by using an inductive structure of solutions of $(\sharp)_k$ revealed in [3].

To give a recursive description of the solutions, we change the variable by setting

$$f(\tau)/\phi_1^{5k}=F(t),$$

where

$$t=\phi_2^5/\phi_1^5=q-5q^2+15q^3-30q^4+40q^5+\cdots$$

is (the reciprocal of) the "Hauptmodul" of $\Gamma_1(5)$, and we consider the equation locally as t a local variable. Then by a routine computation we see that $f(\tau)$ satisfies $(\sharp)_k$ if and only if $F(t)$ satisfies $(\flat)_k$:

$$(\flat)_k \qquad t(t^2+11t-1)F''(t)+\left(\frac{7-11k}{6}t^2+11(1-k)t+\frac{k-5}{6}\right)F'(t) + \frac{k(5k-1)}{6}(t+3)F(t)=0,$$

where $'=d/dt$. Incidentally, an amusing remark here is that the equation

$$(\flat)_{-1} \qquad t(t^2+11t-1)F''(t)+(3t^2+22t-1)F'(t)+(t+3)F(t)=0$$

obtained by setting $k=-1$ in $(\flat)_k$ is exactly the one used in [1] for reconstructing Apéry's irrationality proof of $\zeta(2)$. The original equation $(\sharp)_k$ when $k=-1$ becomes the trivial $f''=0$, but the solutions here are 1 and τ, "universal periods" of elliptic curves. So $(\flat)_{-1}$ is obtained from this trivial equation by rewriting it locally in terms of t-variable.

Now we want to show that $(\flat)_k$ has a polynomial solution $P(t)$ if $k=(6n+1)/5$ ($n\not\equiv 4 \pmod 5$). If $P(t)$ is such a solution, then $\phi_1^{5k}P(\phi_2^5/\phi_1^5)$ and $\phi_2^{5k}P(-\phi_1^5/\phi_2^5)$ are the solutions to $(\sharp)_k$. The second one is a solution because $SL(2,\mathbf{Z})$ acts on the solution space and the action of $\begin{pmatrix} -3 & -5 \\ 5 & 8\end{pmatrix}$ is $\phi_1\mapsto\phi_2,\ \phi_2\mapsto -\phi_1$ (we can check this by using the transformation formulas of ϕ_1 and ϕ_1, see [2]).

Proposition. *For $0 \le n \le 8,\ n \ne 4$, put*

$$
\begin{aligned}
P_0(t) &= 1,\\
P_1(t) &= 1+7t,\\
P_2(t) &= 1+39t-26t^2,\\
P_3(t) &= 1+171t+247t^2-57t^3,\\
P_5(t) &= 1-465t-10385t^2-2945t^3-8370t^4+682t^5,\\
P_6(t) &= 1-333t-17390t^2-54390t^3+26640t^4-64158t^5+3774t^6,\\
P_7(t) &= 1-301t-36421t^2-310245t^3+10535t^4-422303t^5+283843t^6\\
&\quad -12857t^7,\\
P_8(t) &= 1-294t-101528t^2-1798692t^3-2747430t^4-387933t^5\\
&\quad -2086028t^6+740544t^7-26999t^8.
\end{aligned}
$$

For $n \ge 10$, $n \not\equiv 4 \pmod 5$, define $P_n(t)$ recursively by

$$
\begin{aligned}
P_n(t) &= (1+t^2)(1-522t-10006t^2+522t^3+t^4)P_{n-5}(t) \\
&\quad +12\frac{(6n-29)(6n-49)}{(n-4)(n-9)}t(1-11t-t^2)^5 P_{n-10}(t).
\end{aligned}
\tag{1}
$$

Then $P_n(t)$ is a solution of $(\flat)_{(6n+1)/5}$ for all $n \ge 0,\ n \not\equiv 4 \pmod 5$.

Proof. We prove Proposition by induction on n. One may notice that the proof is essentially a translation of Proposition 1 and Lemma in [3].

It is straightforward to check that each $P_n(t)$ for $n \le 8, n \ne 4$ satisfies $(\flat)_{(6n+1)/5}$. Suppose that $P_{n-5}(t)$ and $P_{n-10}(t)$ satisfy $(\flat)_{(6(n-5)+1)/5}$ and $(\flat)_{(6(n-10)+1)/5}$ respectively. If we compute the left-hand side of $(\flat)_{(6n+1)/5}$ for $F(t) = P_n(t)$ by substituting the definition (1) of $P_n(t)$ in terms of $P_{n-5}(t)$ and $P_{n-10}(t)$, and using the induction hypothesis $(\flat)_{(6(n-5)+1)/5}$ and $(\flat)_{(6(n-10)+1)/5}$, we see that $P_n(t)$ satisfies $(\flat)_{(6n+1)/5}$ if and only if the identity

$$
\begin{aligned}
&12(36n^2-468n+1421)(t^2+11t-1)^4 P_{n-10}(t)\\
&\quad = 5(n-9)(t^4-228t^3+494t^2+228t+1)P'_{n-5}(t)\\
&\qquad -(6n^2-83n+261)(t^3-171t^2+247t+57)P_{n-5}(t)
\end{aligned}
\tag{2}
$$

holds. We prove this also by induction on n. Suppose $P_{n-5}(t)$ and $P_{n-10}(t)$ satisfy (2). We want to show the corresponding identity for n being replaced by $n+5$. By replacing $P_n(t)$ by the right-hand side of (1) and then replacing $P_{n-10}(t)$ by the right-hand side of (2) divided by the coefficient $12(36n^2-468n+1421)(t^2+11t-1)^4$ (thus expressing everything by $P_{n-5}(t)$ and its derivatives), we obtain a multiple of the left-hand side of the differential equation $(\flat)_{(6(n-5)+1)/5}$ satisfied by $P_{n-5}(t)$, which vanishes by the induction hypothesis. This concludes the proof of the proposition and hence the theorem is proved.

3 Reduction modulo prime

In this section, we present some observation about reduction modulo a prime p of our polynomials $P_n(t)$ as a conjecture.

Let

$$j(t) = (1 + 228t + 494t^2 - 228t^3 + t^4)^3/t(1 - 11t - t^2)^5$$

be the elliptic modular j-invariant expressed in terms of $t = \phi_2^5/\phi_1^5$.

Conjecture. 1) *Let $p \neq 5$ be a prime. Then $P_{p-1}(t) \bmod p$ is a "supersingular t-polynomial", i.e., it is equal to $\prod_{t_0 \in \bar{\mathbf{F}}_p}(t - t_0)$ where t_0 runs through such value that the corresponding elliptic curve with j-invariant $j(t_0)$ is supersingular.*

2) *For $p \geq 7$, the degrees of irreducible factors of $P_{p-1}(t) \bmod p$ are as follows:*

(i) If $p \equiv 1 \bmod 5$, all irreducible factors have degree 2.

(ii) If $p \equiv 3, 7 \bmod 20$, one factor has degree 2 and all the others have degree 4.

(iii) If $p \equiv 13, 17 \bmod 20$, all irreducible factors have degree 4.

(iv) If $p \equiv 4 \bmod 5$, then there are h linear factors and $(p - 1 - h)/2$ quadratic factors, where

$$h = \textit{the class number of the imaginary quadratic field } \mathbf{Q}(\sqrt{-p}) \times \begin{cases} 2 & \textit{if } p \equiv 1 \bmod 4 \\ 8 & \textit{if } p \equiv 3 \bmod 8 \\ 4 & \textit{if } p \equiv 7 \bmod 8 \end{cases}$$

At least the first part of the conjecture should be proven by looking at the Hasse invariant of a family of elliptic curves corresponding to $\Gamma_1(5)$, but we have not worked out this.

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Graduate School of Mathematics 33, Kyushu University,
Fukuoka 812-8581, Japan.
mkaneko@math.kyushu.u-ac.jp